Infinite Symmetry of the One-Dimensional System with Inverse Square Interactions*

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Abstract

We consider the one-dimensional quantum particle system with inverse square interactions, and reveal the underlying infinite symmetry. We construct explicitly the $\mathcal{W}$-operators to show that the system represents the $w_{1+\infty}$-algebra. Both the Calogero-type ($1/x^2$-interactions) and Sutherland-type ($1/\sin^2 x$-interactions) models are included in these operators : the former is the spin-1 operators and the latter the zero-mode operators. The symmetry of the Calogero-Moser-type (Calogero-type model confined in the harmonic potential) is also discussed.

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1 Introduction

In this paper we report the symmetry of the quantum N-particle system with inverse square interactions. This system is first introduced by Calogero [1], and further studied by Sutherland [2], Moser [3], and others (see, for review, [4]). The symmetry of this system, however, has not been clarified yet. Based on the quantum inverse scattering method we shall find the symmetry of the long-range interaction system.

The Hamiltonian, which we consider, is

$$\begin{aligned}\mathcal{H}_C &= \sum_{j=1}^{N} p_j^2 + \sum_{1\leq j<k\leq N} V(x_j - x_k) \\ &= \sum_j p_j^2 + \sum_{j<k} \frac{2(a^2-a)}{(x_j-x_k)^2},\end{aligned} \tag{1.1}$$

where p_j is the momentum operator satisfying the canonical commutation relation,

$$\left[x_j, p_k\right] \equiv x_j p_k - p_k x_j = i\delta_{jk}. \tag{1.2}$$

We call (1.1) as the Calogero-type model. Well known is the fact that the ground state, Ψ_{G}, of this model is written as the Jastrow-type wave function,

$$\Psi_{\mathrm{G}} = \prod_{j<k} (x_j - x_k)^a. \tag{1.3}$$

The Calogero-type model on a line is integrable and remains so under the periodic boundary condition. When the periodic boundary condition is imposed, the potential $V(x)$ is changed into

$$V(x) = \sum_{n=-\infty}^{\infty} (x + n\pi)^{-2} = \sin^{-2} x, \tag{1.4}$$

where we set the size of the system to be π for simplicity. Then the Hamiltonian becomes

$$\mathcal{H}_S = \sum_{j=1}^{N} p_j^2 + \sum_{j<k} \frac{2(a^2-a)}{\sin^2(x_j - x_k)}, \tag{1.5}$$

which we call the Sutherland-type model.

One of the extensions is the addition of the harmonic potential,

$$\mathcal{H}_{CM} = \sum_j (p_j^2 + \omega^2 x_j^2) + \sum_{j\neq k} \frac{a^2-a}{(x_j-x_k)^2}, \tag{1.6}$$

which is referred to as the Calogero-Moser type model. The energy spectrum is gapful to form the Landau level while the Sutherland-type is gapless.

This paper is organized as follows. In section 2 we construct the conserved operators and give extended Lax equations. In section 3 we show that the $w_{1+\infty}$ algebra is realized. The Calogero-type and the Sutherland-type models appear as one of these operators. In section 4 the Calogero-Moser type model is also discussed. Section 5 is devoted to the conclusions and the discussions.

2 Lax Equation

We introduce the Lax operators L and M_2, which are $N \times N$ operator-valued matrices :

$$L_{jk} = \delta_{jk} \cdot p_j + (1-\delta_{jk}) \cdot \frac{ia}{x_j - x_k}, \tag{2.1}$$

$$(M_2)_{jk} = \delta_{jk} \cdot \sum_{l \neq j} \frac{2a}{(x_j - x_l)^2} - (1-\delta_{jk}) \cdot \frac{2a}{(x_j - x_k)^2}. \tag{2.2}$$

Remark that the operator M_2 satisfies "the sum-to-zero condition",

$$\sum_{l=1}^{N} (M_2)_{jl} = \sum_{l=1}^{N} (M_2)_{lk} = 0. \tag{2.3}$$

This condition plays a crucial role throughout this paper. In addition to the Lax operators, we define "the coordinate operator" as

$$X = \mathrm{diag}(ix_1, ix_2, \cdots, ix_N). \tag{2.4}$$

The imaginary unit i is inclueded for our convenience. These three operator-valued matrices satisfy the following Heisenberg equations of motion :

$$\left[\mathcal{H}_C, L\right] = \left[L, M_2\right], \tag{2.5}$$

$$\left[\mathcal{H}_C, X\right] = \left[X, M_2\right] + 2L. \tag{2.6}$$

The former is the Lax equation, and the latter "the additional equation".

With the Lax equations (2.5) and the sum-to-zero condition (2.3), we can prove that a set of the conserved operators $\mathcal{I}_n$ is constructed from the Lax operator L as

$$\mathcal{I}_n = \sum_{j,k} (L^n)_{jk}. \tag{2.7}$$

For instance, some of them are

$$\begin{aligned}
\mathcal{I}_1 &= \sum_j p_j \equiv \text{total momentum}, \\
\mathcal{I}_2 &= \mathcal{H}_C, \\
\mathcal{I}_3 &= \sum_j p_j^3 + \sum_{j \neq k} \frac{3(a^2 - a)}{(x_j - x_k)^2} p_j.
\end{aligned}$$

For further discussions, it is important to notice that the extended Lax equations and the additional equations are satisfied for $\mathcal{I}_n$;

$$\left[\mathcal{I}_n, L\right] = \left[L, M_n\right], \tag{2.8}$$

$$\left[\mathcal{I}_n, X\right] = \left[X, M_n\right] + nL^{n-1}, \tag{2.9}$$

where M_n are $N \times N$ matrices satisfying the sum-to-zero condition,

$$\sum_l (M_n)_{jl} = \sum_l (M_n)_{lk} = 0. \tag{2.10}$$

These equations are proved for the classical [6, 7] and quantum [8] cases. Equation (2.8) leads to the involutiveness of the conserved operators,

$$\Big[\mathcal{I}_n, \mathcal{I}_m\Big] = 0, \tag{2.11}$$

which supports the integrability of the Calogero-type model in the Liouville sense.

3 W Algebra

The conserved quantities can be extended to the "higher spin" using the coordinate matrix X. We define the $\mathcal{W}$-operators recursively by

$$\begin{aligned} \mathcal{W}_n^{(s)} &= \frac{1}{2(n+s)}\Big[\sum_j x_j^2, \mathcal{W}_{n+2}^{(s-1)}\Big], \\ \mathcal{W}_n^{(1)} &\equiv \mathcal{I}_n. \end{aligned} \tag{3.1}$$

The superscript s indicates spin. We note that the $\mathcal{W}$-operators have the form,

$$\mathcal{W}_n^{(s)} = \sum_j (ix_j)^{s-1} p_j^{n+s-1} + \cdots, \tag{3.2}$$

and are only defined for $n \geq -s+1$.

These $\mathcal{W}$-operators constitute the $w_{1+\infty}$ algebra [11] (strictly speaking, $w_{1+\infty}^{+}$ algebra, a subalgebra of $w_{1+\infty}$ algebra),

$$\Big[\mathcal{W}_n^{(s)}, \mathcal{W}_m^{(s')}\Big] = \Big((s'-1)n - (s-1)m\Big)\mathcal{W}_{n+m}^{(s+s'-2)} + (\text{lower terms}), \tag{3.3}$$

We write some of the $\mathcal{W}$-operators explicitly :

$$\mathcal{W}_{-1}^{(2)} = \sum_j ix_j, \qquad \mathcal{W}_0^{(2)} = \sum_j ix_j p_j + \frac{N}{2}, \tag{3.4}$$

$$\begin{aligned} \mathcal{W}_{-2}^{(3)} &= \sum_j (ix_j)^2, \qquad \mathcal{W}_{-1}^{(3)} = \sum_j (-x_j^2 p_j + ix_j), \\ \mathcal{W}_0^{(3)} &= \sum_j (-x_j^2 p_j^2 + 2ix_j p_j) + \sum_{j \neq k} (a^2 - a)\frac{x_j x_k}{(x_j - x_k)^2} + \frac{N}{6}\Big(3 + 2a(1-a)(N-1)\Big) \end{aligned} \tag{3.5}$$

One notices (3.3) that not only spin-1 operators $\mathcal{W}_n^{(1)}$ but also the zero-mode operators $\mathcal{W}_0^{(s)}$ constitute the involutive sets. The significance of the zero-mode operators is found by changing the variables. We introduce as z_j [10],

$$x_j \equiv \exp(2iz_j). \tag{3.6}$$

Note that z_j satisfies the periodic boundary condition (recall that the system size is π). By substituting (3.6) into (3.4) and (3.5), we have

$$\mathcal{W}_0^{(2)} = \frac{1}{2}\sum_j(-i\partial_j) + \frac{N}{2}, \tag{3.7}$$

$$\begin{aligned}\mathcal{W}_0^{(3)} &= \frac{1}{4}\Big(-\sum_j \partial_j^2 + \sum_{j\neq k}\frac{a^2-a}{\sin^2(z_j-z_k)}\Big) + \frac{1}{2}\sum_j(-i\partial_j) \\ &\quad + \frac{N}{6}\Big(3+2a(1-a)(N-1)\Big),\end{aligned} \tag{3.8}$$

where ∂_j denotes $\partial/\partial z_j$. This result shows that the zero-mode operators $\mathcal{W}_0^{(s)}$ are nothing but the conserved operators of the Sutherland-type model $\mathcal{H}_S$.

4 Calogero Model in the Harmonic Potential

An interesting extension is the Calogero-type model confined in the harmonic potential,

$$\mathcal{H}_{\rm CM} = \sum_j p_j^2 + \sum_{j\neq k}\frac{a^2-a}{(x_j-x_k)^2} + \sum_j \omega^2 x_j^2, \tag{4.1}$$

where the subscript CM means the Calogero-Moser model. In this case the Lax equation is modified into

$$\Big[\mathcal{H}_{CM}, L\pm\omega X\Big] = \Big[L\pm\omega X, M_2\Big] \pm 2\omega(L\pm\omega X). \tag{4.2}$$

The W operators are defined by

$$\widehat{W}_n^{(s)} = \sum_{j,k}(L+\omega X)^{s-n-1}\cdot(L-\omega X)^{s+n-1}, \tag{4.3}$$

which constitute the w_∞ algebra [5],

$$\Big[\widehat{W}_n^{(s)}, \widehat{W}_m^{(s')}\Big] = 4\omega\Big((s'-1)n-(s-1)m\Big)\widehat{W}_{n+m}^{(s+s'-2)} + (\text{ lower terms }). \tag{4.4}$$

In these operators, $\widehat{W}_n^{(s)}$ is defined for $-s+1\leq n\leq s-1$ for n and s being integer or half-odd-integer. It is important to notice that the Hamiltonian $\mathcal{H}_{CM}$ is $\widehat{W}_0^{(2)}$, and that operators satisfy

$$\Big[\mathcal{H}_{CM}, \widehat{W}_n^{(s)}\Big] = -4n\omega\widehat{W}_n^{(s)}. \tag{4.5}$$

These relations suggest that the $\widehat{W}_n^{(s)}$ can be regarded as the creation and annihilation operators of the Calogero-Moser type Hamiltonian.

5 Conclusion

We have revealed the symmetry of the quantum one-dimensional models with inverse square interactions. From the viewpoint of the $w_{1+\infty}$ algebra, both of the Calogero and Sutherland type models can be understood as follows ;

	Calogero type	Sutherland type
Conserved operators	Spin-1	Zero-mode
	$\mathcal{W}_n^{(1)}$	$\mathcal{W}_0^{(s)}$

At the moment, it is still unclear whether the system with the $\wp$-function interactions has such symmetry or not.

The Calogero model can be generalized to $su(\nu)$ case. Each particle has an $su(\nu)$ spin as an internal degree of freedom. The Hamiltonian is

$$\mathcal{H}_{su(\nu)} = \sum_j^N p_j^2 + \sum_{j \neq k} \frac{a^2 - aP_{jk}}{(x_j - x_k)^2}, \tag{5.1}$$

where P is the permutation operator in the $su(\nu)$ spin space. For $su(2)$ case, it is expressed in terms of the Pauli spin matrices,

$$P_{jk} = \frac{1}{2}\Big(1 + \sigma_j \cdot \sigma_k\Big). \tag{5.2}$$

As one expect, the $w_{1+\infty}$ symmetry is extended to the coloured symmetry, the $su(\nu)$-$w_{1+\infty}$ algebra [11].

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